

# A FINITE QUANTUM SYMMETRY OF $M(3, \mathbb{C})$

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## Abstract

The 27-dimensional Hopf algebra  $A(F)$ , defined by the exact sequence of quantum groups  $A(SL(2, \mathbb{C})) \xrightarrow{Fr} A(SL_q(2)) \xrightarrow{\pi_F} A(F)$ ,  $q = e^{\frac{2\pi i}{3}}$ , is studied as a finite quantum group symmetry of the matrix algebra  $M(3, \mathbb{C})$ , describing the color sector of Alain Connes' formulation of the Standard Model. The duality with the Hopf algebra  $\mathcal{H}$ , investigated in a recent work by Robert Coquereaux, is established and used to define a representation of  $\mathcal{H}$  on  $M(3, \mathbb{C})$  and two commuting representations of  $\mathcal{H}$  on  $A(F)$ .

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# Introduction

In recent years there has been a growing interest in establishing the links between noncommutative geometry and quantum groups in analogy to the important relations between classical geometry and group theory. An interesting territory to test their possible interplay is Connes' formulation of the Standard Model of elementary particles (for its latest version, including also the gravity, see [ChC]). In the Standard Model, which is extraordinarily successful, there remain still some fundamental open questions. It is tempting to investigate if for resolving them some new symmetry of the quantum group type (perhaps finite) could be helpful. This seems quite a natural question having at a disposal a noncommutative formulation of the Standard Model.

At the end of Ref. [C-A] the sequence  $1 \rightarrow F \rightarrow SU_q(2) \rightarrow SU(2) \rightarrow 1$ , where  $q = e^{\frac{2\pi i}{3}}$  and  $F$  is a finite quantum group, has been suggested in relation to the Standard Model or its hypothetical extensions to higher energy regimes. In the paper [DHS] a possible interpretation of this sequence is given both in the sense of exact sequence of Hopf algebras and in the language of principal quantum bundles (Hopf-Galois extensions). More precisely, for  $q^3 = 1$ , there is an exact sequence of Hopf algebras  $A(SL(2, \mathbb{C})) \xrightarrow{Fr} A(SL_q(2)) \xrightarrow{\pi_F} A(F)$ , where  $A(F)$  is a finite dimensional quotient Hopf algebra of  $A(SL_q(2))$ . It is furthermore shown that  $A(SL_q(2))$  is a faithfully flat  $A(F)$ -extension of  $A(SL(2, \mathbb{C}))$ .

In the present communication, we elaborate more on  $A(F)$ , focusing our attention on its possible role as a quantum symmetry of the algebra  $M(3, \mathbb{C})$ , which is a part of the algebra  $\mathcal{A} = M(3, \mathbb{C}) \oplus \mathbb{H} \oplus \mathbb{C}$  used by Connes for his formulation of the Standard Model. In section 1, we recall the basic results from [DHS]. The first part of this section contains noncommutative generalizations of some geometrical structures, and serves to give a more mathematical framework to the main object of our interest,  $A(F)$ , which is introduced in the second part. In section 2, we discuss the coaction of  $A(F)$  on  $M(3, \mathbb{C})$ , the color sector of  $\mathcal{A}$ , and possible extensions to the other two sectors. In section 3, we describe the parallel work of [C-R] in the framework of universal enveloping algebras and we establish the Hopf duality between the algebra  $\mathcal{H}$ , defined therein, and  $A(F)$ . In section 4, using this duality we exhibit a representation of  $\mathcal{H}$  on  $M(3, \mathbb{C})$  that can be extended to  $\mathcal{A}$ . In this representation the generator  $K$  of  $\mathcal{H}$  acts as automorphism while the generators  $X_{\pm}$  act as *twisted* derivations. The automorphism is clearly inner and it turns out that

the twisted derivations can be also expressed as a sort of internal operations ( $\mathbb{Z}_3$ -graded bracket with some elements  $\tilde{X}_\pm$ ). In section 5, we give two commuting representations of  $\mathcal{H}$  on  $A(F)$ . (The explicit results are tabulated in the Appendix). Furthermore, we use the notion of integrals *in* and *on* a Hopf algebra to say more about the algebraic and coalgebraic properties of  $\mathcal{H}$  and  $A(F)$ .

We consider the present contribution as a step in the direction of achieving physically significant statements about a possible quantum symmetry behind the Standard Model. In this respect it is particularly interesting to answer the question of how to implement the finite quantum symmetry  $A(F)$  on the level of representation spaces of  $\mathcal{A}$  (which describe matter fields), and to verify if the generators of  $\mathcal{H}$  implemented as operators preserve the action integral in some sense. These questions shall be investigated in our future work.

## 1 Preliminaries

Let us recall that, given a sequence of groups and group homomorphisms  $G \rightarrow G' \rightarrow G''$ , one can consider the (dual) sequence  $B \xrightarrow{j} P \xrightarrow{\pi} H$  of Hopf algebras of functions and of Hopf algebra morphisms (pull back of the mappings reverses the arrows). The exactness of the sequence of groups translates then into the definition [PW, S1] of *exact* sequence of Hopf algebras. This definition requires that  $j$  has to be injective,  $H = P/Pj(B^+)P$ , where  $B^+$  denotes the kernel of the counit map of  $B$ , and that  $\pi$  is the canonical surjection. It applies directly to the noncommutative case.

In the classical (commutative) case an exact sequence of groups is equivalent to a principal fibre bundle with principal space  $G'$ , base space  $G'/G = G''$  and structure group  $G$ . In the noncommutative case this is no longer automatically true, and further conditions must be imposed in order to achieve the equivalence. Let us comment on this in more detail.

An exact sequence of Hopf algebras is called *strictly exact* [S1] iff  $P$  is a right faithfully flat module over  $j(B)$ , and  $j(B)$  is a normal Hopf subalgebra of  $P$ , i.e.,  $p_{(1)}j(B)S(p_{(2)}) \cup S(p_{(1)})j(B)p_{(2)} \subseteq j(B)$  for any  $p \in P$ , where the Sweedler notation is used for the coproduct and  $S$  is the antipode.

Next, a suitable dualization of (some of) the properties of a principal fibre bundle is

achieved via the notion of Hopf-Galois extension [KT]. If  $H$  is a Hopf algebra,  $P$  is a right  $H$ -comodule algebra and  $B = P^{coH}$  (the space of coinvariants of the coaction), we say that  $P$  is a (right) *Hopf-Galois  $H$ -extension* iff the canonical left  $P$ -module right  $H$ -comodule map

$$(m_P \otimes id) \circ (id \otimes_B \Delta_R) : P \otimes_B P \longrightarrow P \otimes H$$

is bijective. In addition, we say that it is *faithfully flat* iff  $P$  is faithfully flat (right)  $B$ -module.

In the classical case,  $B$ , the algebra of functions on the quotient space, is identified with the subalgebra of functions on the principal space that are constant on the fibres and the canonical map is just the pull-back of the map  $X \times G \rightarrow X \times_M X$  given by  $(x, g) \mapsto (x, xg)$ , whose bijectivity means that the action is free and transitive on the fibres. A particular kind of Hopf-Galois extensions is given by the *cleft* ones: an  $H$ -Galois extension is called *cleft* iff there exists a unital, convolution invertible, linear map  $\Phi : H \rightarrow P$  satisfying  $\Delta_R \circ \Phi = (\Phi \otimes id) \circ \Delta$ .

In the datum of an exact sequence of Hopf algebras we have, in particular, a quotient Hopf algebra  $H$ , which coacts in a natural way via *push out* ( $\Delta_R = (Id \otimes \pi) \circ \Delta$ ) on  $P$ , and a Hopf subalgebra  $j(B)$ . It remains to be verified whether  $j(B)$  coincides with the space of coinvariants  $P^{coH}$  and whether  $P$  is faithfully flat Hopf-Galois  $H$ -extension of  $j(B)$ . It turns out [S1] that for *strictly exact* sequences of Hopf algebras this is indeed the case.

Now we pass to the case of our interest. We recall that  $A(SL_q(2))$  is a complex Hopf algebra generated by  $T_{ij} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , satisfying the following relations:

$$\begin{aligned} ab &= qba, \quad ac = qca, \quad bd = qdb, \quad bc = cb, \quad cd = qdc, \\ ad - da &= (q - q^{-1})bc, \quad ad - qbc = da - q^{-1}bc = 1. \end{aligned} \tag{1}$$

The comultiplication  $\Delta$ , counit  $\varepsilon$ , and antipode  $S$  are

$$\begin{aligned} \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\ \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}. \end{aligned} \tag{2}$$

As a complex vector space,  $A(SL_q(2))$  has a basis  $a^i b^j c^k$ ;  $i, j, k \geq 0$  and  $b^i c^j d^k$ ;  $i, j \geq 0, k > 0$ .

From now on, unless stated differently, we set the parameter  $q = e^{\frac{2\pi i}{3}}$ .

Let  $\bar{T}_{ij}$  denote the generators of undeformed  $A(SL(2, \mathbb{C}))$ .

In [DHS] the following commutative diagram of algebras and algebra homomorphisms has been introduced:

$$\begin{array}{ccc}
A(\mathbb{C}^2) & \xrightarrow{\rho} & A(\mathbb{C}^2) \otimes A(SL(2, \mathbb{C})) \\
fr \downarrow & & \downarrow fr \otimes Fr \\
A(\mathbb{C}_q^2) & \xrightarrow{\rho_q} & A(\mathbb{C}_q^2) \otimes A(SL_q(2, \mathbb{C})) \\
\pi_M \downarrow & & \downarrow \pi_M \otimes \pi_F \\
M(3, \mathbb{C}) & \xrightarrow{\rho_F} & M(3, \mathbb{C}) \otimes A(F) .
\end{array} \tag{3}$$

We explain now the various ingredients of (3).

First,  $A(\mathbb{C}^2) = \mathbb{C}[\bar{x}, \bar{y}]$  is the algebra of polynomials on  $\mathbb{C}^2$ ,  $A(\mathbb{C}_q^2)$  is the algebra of the quantum plane, i.e. the free algebra generated by  $x$  and  $y$  modulo the relation  $xy = qyx$ , and  $fr$  is the injection given by  $fr(\bar{x}) = x^3$ ,  $fr(\bar{y}) = y^3$ . The map  $\pi_M$  is the composition of the canonical projection  $x \mapsto \tilde{x}$ ,  $y \mapsto \tilde{y}$ , from the algebra  $A(\mathbb{C}_q^2)$  to the algebra  $A(\mathbb{C}_q^2)$  modulo the relations  $x^3 = 1$  and  $y^3 = 1$ , with the map

$$\tilde{x} \mapsto \begin{pmatrix} 0 & 1_{n-1} \\ 1 & 0 \end{pmatrix}, \quad \tilde{y} \mapsto \text{diag}(1, q, \dots, q^{n-1}), \tag{4}$$

identifying (for  $n = 3$ ) the latter algebra with the algebra of matrices  $M(n, \mathbb{C})$  (see Section IV.D.15 of [W-H]).

Next, the Hopf algebra injection  $Fr : A(SL(2, \mathbb{C})) \ni \bar{T}_{ij} \mapsto (T_{ij})^3 \in A(SL_q(2))$ ,  $i, j = 1, 2$ , is the so-called *Frobenius* mapping. Moreover,  $A(F)$  is the finite dimensional quotient Hopf algebra of  $A(SL_q(2))$  modulo the relations

$$a^3 = 1 = d^3, \quad b^3 = 0 = c^3, \tag{5}$$

and  $\pi_F$  is the canonical projection.

Finally,  $\rho$ ,  $\rho_q$  and  $\rho_F$  are the natural right coactions on  $A(\mathbb{C}^2)$ ,  $A(\mathbb{C}_q^2)$ , and  $M(3, \mathbb{C})$ , respectively, given symbolically by  $e_i \mapsto \sum_{j=1,2} e_j \otimes M_{ji}$ ,  $i = 1, 2$ .

In [DHS] the following facts are shown:

- i) the sequence  $A(SL(2, \mathbb{C})) \xrightarrow{Fr} A(SL_q(2)) \xrightarrow{\pi_F} A(F)$  is strictly exact
- ii)  $A(SL_q(2))$  is a (right, faithfully flat) Hopf-Galois  $A(F)$ -extension of  $Fr(A(SL(2, \mathbb{C})))$
- iii)  $A(F)$  is a 27-dimensional complex vector space with a basis  $\tilde{a}^p \tilde{b}^r \tilde{c}^s$ ;  $p, r, s \in \{0, 1, 2\}$ , where  $\tilde{a} = \pi_F(a)$ , etc.

iv)  $A(F)$  has a faithful representation <sup>1</sup>  $\varrho$

$$\varrho(\tilde{a}) = \mathbf{J} \otimes \mathbf{1}_3 \otimes \mathbf{1}_3, \quad \varrho(\tilde{b}) = \mathbf{Q} \otimes \mathbf{N} \otimes \mathbf{1}_3, \quad \varrho(\tilde{c}) = \mathbf{Q} \otimes \mathbf{1}_3 \otimes \mathbf{N}, \quad (6)$$

where

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (7)$$

v)  $fr(A(\mathbb{C}^2)) = A(\mathbb{C}_q^2)^{coA(F)}$ , ie. “Frobenius-like” map  $fr$  allows to identify  $A(\mathbb{C}^2)$  with the subalgebra of  $(id \otimes \pi_F) \circ \rho_q$ -coinvariants of  $A(\mathbb{C}_q^2)$ .

## 2 Quantum symmetries of $M(3, \mathbb{C})$

First we elaborate more on  $F$  as a quantum-group symmetry of  $M(3, \mathbb{C})$  — a direct summand of Connes’ algebra for the Standard Model.

It is easy to see that the subalgebra of coinvariants of  $M(3, \mathbb{C})$  under the coaction of  $A(F)$  is one dimensional:  $M(3, \mathbb{C})^{coA(F)} = \mathbb{C}$ . This leads us to think of  $M(3, \mathbb{C})$  as a quantum homogeneous space of  $A(F)$ . Notice, however, that  $M(3, \mathbb{C})$  is *not* an embeddable  $A(F)$ -space in the sense of [B-T], because there does not exist an algebra injection  $i : M(3, \mathbb{C}) \rightarrow A(F)$  (in particular, the algebra  $M(3, \mathbb{C})$  has no characters).

The classical subgroup of  $F$  is given by the set of characters of  $A(F)$ , i.e. non zero algebra morphisms  $\chi : A(F) \rightarrow \mathbb{C}$ , endowed with the convolution product  $(\chi \cdot \psi)(u) = (\chi \otimes \psi) \circ \Delta(u)$ . It is easy to see that there are only three characters  $\chi_i$ ,  $i = 0, 1, 2$ . Their action on generators of  $A(F)$  is given by  $\chi_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^i & 0 \\ 0 & q^{2i} \end{pmatrix}$ . The classical subgroup of  $F$  is then isomorphic to  $\mathbb{Z}_3$ . The Hopf algebra  $A(\mathbb{Z}_3)$  appears as a quotient of  $A(F)$  by the ideal generated by  $\tilde{b}$ ,  $\tilde{c}$ . Notice that this ideal is the intersection of the kernels of the characters.

Next,  $A(\mathbb{Z}_3)$  coacts on  $M(3, \mathbb{C})$  via push-out. Using the basis of  $M(3, \mathbb{C})$  given by  $\tilde{x}^r \tilde{y}^s$ ;  $r, s \in \{0, 1, 2\}$ , one has  $M(3, \mathbb{C})^{coA(\mathbb{Z}_3)} = \text{span}_{\mathbb{C}}\{1, \tilde{x}\tilde{y}, \tilde{x}^2\tilde{y}^2\} \cong \mathbb{C}^3$ . It turns out that the extension  $(M(3, \mathbb{C}), \mathbb{C}^3, A(\mathbb{Z}_3))$  is a cleft Hopf-Galois extension, with the unital, right covariant, convolution invertible map  $\Phi : A(\mathbb{Z}_3) \rightarrow M(3, \mathbb{C})$  being given by

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<sup>1</sup>For a representation in terms of Grassman variables see [S-A].

$\Phi(\tilde{a}) = \tilde{x}$ ,  $\Phi(\tilde{a}^2) = \tilde{x}^2$ . Being  $\mathbb{C}^3$  a commutative subalgebra, it also follows from [S2] that the extension is faithfully flat.<sup>2</sup>

Making use of the coaction of  $A(\mathbb{Z}_3)$  on  $M(3, \mathbb{C})$  and of the characters of  $A(\mathbb{Z}_3)$ , we can define three algebra endomorphisms of  $M(3, \mathbb{C})$  by

$$F_i = (Id \otimes \chi_i) \circ \Delta_R, \quad i = 0, 1, 2. \quad (8)$$

Explicitly one has:

$$F_i(\tilde{x}) = q^i \tilde{x}, \quad F_i(\tilde{y}) = q^{2i} \tilde{y}. \quad (9)$$

The mapping  $\chi_i \mapsto F_i$  identifies  $\mathbb{Z}_3$  as a subgroup of the group of algebra automorphisms of  $M(3, \mathbb{C})$ , which is  $SU(3)/\mathbb{Z}_3^{\text{diag}}$ , where  $\mathbb{Z}_3^{\text{diag}} = \{\mathbf{1}_3, q\mathbf{1}_3, q^2\mathbf{1}_3\}$ . More precisely, eg. the generator  $\chi_1$  of  $\mathbb{Z}_3$  corresponds to an inner automorphism via the adjoint action (of the  $\mathbb{Z}_3^{\text{diag}}$ -class) of the matrix

$$\mathbf{U}_1 = \tilde{x}^2 \tilde{y}^2 = \begin{pmatrix} 0 & 0 & q \\ 1 & 0 & 0 \\ 0 & q^2 & 0 \end{pmatrix}. \quad (10)$$

Hence the quantum finite symmetry  $A(F)$  has  $\mathbb{Z}_3$  as an overlap with the classical symmetry group  $SU(3)/\mathbb{Z}_3^{\text{diag}}$  of  $M(3, \mathbb{C})$ .

One may wonder if any nontrivial finite symmetry of the remaining piece  $\mathbb{H} \oplus \mathbb{C}$  of Connes' algebra for the Standard Model can be also obtained in the same spirit. As far as the algebra of quaternions  $\mathbb{H}$  is concerned, it embeds, at least as a real algebra, into  $M(2, \mathbb{C})$  via the mapping

$$u = \alpha + \beta j \mapsto \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}. \quad (11)$$

In terms of the basis of  $M(2, \mathbb{C})$  given by  $\tilde{x}^r \tilde{y}^s$ ;  $r, s \in \{0, 1\}$ , a quaternion  $u$  is then expressed as  $u = \frac{1}{2} ((\alpha + \bar{\alpha})1 + (\beta - \bar{\beta})\tilde{x} + (\alpha - \bar{\alpha})\tilde{y} - (\beta + \bar{\beta})\tilde{x}\tilde{y})$ .

Then, using (4) for  $n = 2$ , we can define a coaction  $\rho_2$  on  $M(2, \mathbb{C})$  of the quotient Hopf algebra of  $A(SL_q(2))$ ,  $q = -1$ , modulo the relations [T-M]

$$a^2 = 1 = d^2, \quad b = 0 = c, \quad (12)$$

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<sup>2</sup> This result holds in general for any root of unity  $q^n = 1$ , so that  $(M(n, \mathbb{C}), \mathbb{C}^n, A(\mathbb{Z}_n))$  is a cleft, faithfully flat, Hopf-Galois extension.

obtaining, however, nothing but a (classical)  $A(\mathbb{Z}_2)$ .

Quaternions are a (real) subcomodule of  $M(2, \mathbb{C})$ , since one has

$$\rho_2(u) = (Re(\alpha) + Re(\beta)j) \otimes 1 + (Im(\alpha) + Im(\beta)j) \otimes \tilde{a} .$$

By composing the coaction with the nontrivial charcter of  $A(\mathbb{Z}_2)$ , we find that the generator of  $\mathbb{Z}_2$  acts on  $M(2, \mathbb{C})$  as the inversion  $\tilde{x} \mapsto -\tilde{x}$ ,  $\tilde{y} \mapsto -\tilde{y}$ , ie. via an inner automorphism by

$$\mathbf{U} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \quad (13)$$

This action preserves  $\mathbb{H}$  and amounts to the complex conjugation of  $\alpha$  and of  $\beta$  in (11).

Next, as far as the algebra  $\mathbb{C} \equiv M(1, \mathbb{C})$  is concerned, it leads, obviously, to a trivial group  $\{e\}$ .

We remark also that, since we can embed  $M(3, \mathbb{C}) \oplus M(2, \mathbb{C}) \oplus \mathbb{C}$  (e.g. in a diagonal way) in  $M(6, \mathbb{C})$ , we have also checked possible quantum symmetries of  $M(6, \mathbb{C})$ . Repeating our construction, there is a quotient Hopf algebra of  $A(SL_q(2))$ ,  $q = e^{2\pi i/6}$ , modulo the relations

$$a^6 = 1 = d^6, \quad b^3 = 0 = c^3. \quad (14)$$

It is, however, not interesting (even neglecting the problem of invariance of the subalgebra  $M(3, \mathbb{C}) \oplus M(2, \mathbb{C}) \oplus \mathbb{C}$ ) for the reason that its dimension is 54, i.e. just the dimension of  $A(F \times \mathbb{Z}_2) = A(F) \otimes A(\mathbb{Z}_2)$ .

(Incidentally, finding the biggest quantum group coacting on a given quantum space seems to be quite an intricate problem).

Obviously, the coaction  $\rho_F$  can be extended to the whole  $\mathcal{A}$  in a trivial way by

$$\hat{\rho}(m + u + l) = m_{(0)} \otimes m_{(1)} + u \otimes 1 + l \otimes 1 , \quad (15)$$

(where we have used Sweedler notation:  $\rho_F(m) = m_{(0)} \otimes m_{(1)} \in M(3, \mathbb{C}) \otimes A(F)$ ), for all  $m \in M(3, \mathbb{C})$ ,  $u \in \mathbb{H}$ ,  $l \in \mathbb{C}$ , so that  $\mathcal{A}$  becomes an  $A(F)$ -comodule algebra.

A less trivial extension should involve a coaction of another Hopf algebra on  $\mathbb{H}$ . So far the only candidate we know is  $A(\mathbb{Z}_2)$ , in which case it gives rise to the right coaction of  $A(F) \otimes A(\mathbb{Z}_2)$

$$\check{\rho}(m + u + l) = m_{(0)} \otimes m_{(1)} \otimes 1 + u_{(0)} \otimes 1 \otimes u_{(1)} + l \otimes 1 \otimes 1 . \quad (16)$$

With this definition,  $\mathcal{A}$  becomes an  $A(F) \otimes A(\mathbb{Z}_2)$ -comodule algebra.



### 3 Duality

In the paper [C-R] a parallel approach to quantum finite symmetries has been discussed in terms of universal enveloping algebras. The 27-dimensional Hopf algebra  $\mathcal{H}$  defined therein is a quotient Hopf algebra of  $U_q(sl(2))$  for  $q^3 = 1$ . It is generated by  $X_+$ ,  $X_-$ ,  $K$ , with the following relations:

$$X_+^3 = X_-^3 = 0, \quad K^3 = 1, \quad KX_\pm = q^{\mp 2}X_\pm K, \quad [X_+, X_-] = \frac{K - K^{-1}}{q^{-1} - q}. \quad (17)$$

(Notice that with respect to the convention used by [C-R] we have changed  $q$  with  $q^{-1}$ , in order to be consistent with our definition of  $A(SL_q(2))$ ).

$\mathcal{H}$  is a Hopf algebra with coproduct, counit and antipode induced by  $U_q(sl(2))$ :

$$\begin{aligned} \Delta(X_+) &= X_+ \otimes 1 + K \otimes X_+, \quad \Delta(X_-) = X_- \otimes K^{-1} + 1 \otimes X_-, \quad \Delta(K) = K \otimes K, \\ \varepsilon(X_+) &= \varepsilon(X_-) = 0, \quad \varepsilon(K) = 1, \\ S(X_+) &= -K^{-1}X_+, \quad S(X_-) = -X_-K, \quad S(K) = K^{-1}. \end{aligned} \quad (18)$$

In [C-R] it is shown that the underlying vector space of  $\mathcal{H}$  has an intriguing splitting as a direct sum of the semisimple subalgebra  $M(3, \mathbb{C}) \oplus M(2, \mathbb{C}) \oplus \mathbb{C}$ , that is very close to Connes' finite algebra  $\mathcal{A}$ , and the radical ideal  $\mathcal{R}$ . The radical  $\mathcal{R}$  is the intersection of kernels of all irreducible representations of  $\mathcal{H}$  and it is isomorphic with the algebra of  $3 \times 3$  matrices of the form

$$\begin{pmatrix} \alpha_{11}\theta_1\theta_2 & \alpha_{12}\theta_1\theta_2 & \beta_{13}\theta_1 + \gamma_{13}\theta_2 \\ \alpha_{21}\theta_1\theta_2 & \alpha_{22}\theta_1\theta_2 & \beta_{23}\theta_1 + \gamma_{23}\theta_2 \\ \beta_{31}\theta_1 + \gamma_{31}\theta_2 & \beta_{32}\theta_1 + \gamma_{32}\theta_2 & \alpha_{33}\theta_1\theta_2 \end{pmatrix}$$

where  $\theta_1, \theta_2$  are two Grassman variables satisfying the relations  $\theta_1^2 = \theta_2^2 = 0$  and  $\theta_1\theta_2 = -\theta_2\theta_1$ .

It is known [K-Ch], and it is explicit in this presentation, that, modulo equivalence, there are only three irreducible representations of  $\mathcal{H}$ , respectively of dimension 1, 2 and 3, and that there are no irreducible representations of dimension greater than 3. From a physical point of view, the basic multiplets of representations of  $\mathcal{H}$  are then, respectively, a singlet with an arbitrary value of hypercharge, null isospin and no color, a doublet of isospin with zero hypercharge and color, and a triplet of color with zero isospin and hypercharge. They cannot describe any physical particle, and using tensor products of basic representations

of  $\mathcal{H}$  via iterating the coproduct doesn't solve the problem, since, as stressed in [C-R], the subalgebra  $M(3, \mathbb{C}) \oplus M(2, \mathbb{C}) \oplus \mathbb{C}$  is not a subcoalgebra, so that the physical "charges" are not additive. Representations of  $\mathcal{H}$  are in general not totally reducible, becoming so only when restricted to the semisimple part. In the sequel, we will give some examples of representations of this kind.

Connes' formulation of the Standard Model uses a 90-dimensional (three families of leptons and quarks are considered, together with their antiparticles) representation of  $\mathcal{A}_F$ , using the embedding of  $\mathbb{H}$  in  $M(2, \mathbb{C})$ , so that it is actually obtained by a representation of the semisimple part of  $\mathcal{H}$  (for a good reference, see [MGV]; see also [LMMS] for problems of such a formulation). Such a representation can be trivially extended to the whole  $\mathcal{H}$ , by setting to 0 the action of the radical  $\mathcal{R}$ . It is an open question whether such an extension is unique.

As regards the relationship between  $\mathcal{H}$  and  $A(F)$ , it is well known that there is a Hopf duality between  $A(SL_q(2))$  and  $U_q(sl(2))$ , in the sense that there exist a bilinear form  $\langle \cdot, \cdot \rangle$  on  $U_q(sl(2)) \times A(SL_q(2))$  such that, for any  $u, v$  in  $U_q(sl(2))$  and for any  $x, y$  in  $A(SL_q(2))$  one has:

$$\begin{aligned} \langle uv, x \rangle &= \langle u, x_{(1)} \rangle \langle v, x_{(2)} \rangle, \quad \langle u, xy \rangle = \langle u_{(1)}, x \rangle \langle u_{(2)}, y \rangle, \\ \langle 1, x \rangle &= \varepsilon(x), \quad \langle u, 1 \rangle = \varepsilon(u), \quad \langle S(u), x \rangle = \langle u, S(x) \rangle. \end{aligned} \quad (19)$$

Explicitly, this pairing makes use of the fundamental representation of  $U_q(sl(2))$  given by:

$$\rho(X_-) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(X_+) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}. \quad (20)$$

Writing for any  $u \in U_q(sl(2))$

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

one sets:

$$\langle u, a \rangle = A(u), \quad \langle u, b \rangle = B(u), \quad \langle u, c \rangle = C(u), \quad \langle u, d \rangle = D(u),$$

and then extends the definition to arbitrary elements of  $A(SL_q(2))$  by using the properties (19).

It turns out that for  $q^n = 1$  the pairing is degenerate and has a huge kernel. In particular,

for  $q^3 = 1$  the kernel contains both the defining ideals of the Hopf algebras  $A(F)$  and  $\mathcal{H}$ , so that the pairing descends to the quotients.

It is convenient to analyse the  $27 \times 27$  matrix of this pairing using as a basis for  $A(F)$  the more symmetric set  $\tilde{a}\tilde{b}^r\tilde{c}^s, \tilde{b}^r\tilde{c}^s, \tilde{d}\tilde{b}^r\tilde{c}^s; r, s \in \{0, 1, 2\}$ . Setting  $\deg(X_-) = \deg(\tilde{b}) = -1$ ,  $\deg(K) = \deg(\tilde{a}) = \deg(\tilde{d}) = 0$ ,  $\deg(X_+) = \deg(\tilde{c}) = 1$ , it turns out that monomials with different total degree are orthogonal, generating a block diagonal matrix with five diagonal blocks. In table 1 of the Appendix we present those non vanishing blocks.

Now, the determinant of our  $27 \times 27$  matrix is given by the product of the determinants of nine  $3 \times 3$  subblocks on the diagonal. It is easy to convince oneself, by looking at the linear independence of the rows (or the columns) of these sub-blocks, that the determinant is different from 0, so that the pairing between  $\mathcal{H}$  and  $A(F)$  is not degenerate. We are thus in a position to state that  $\mathcal{H}$  and  $A(F)$  are dual Hopf algebras.

## 4 A representation of $\mathcal{H}$ on $M(3, \mathbb{C})$

Having exhibited the duality between  $\mathcal{H}$  and  $A(F)$ , since  $M(3, \mathbb{C})$  is a right  $A(F)$ -comodule algebra via the coaction  $\rho_F$ , it becomes a left  $\mathcal{H}$ -module algebra, in the sense that there is a representation (left action) of  $\mathcal{H}$  on  $M(3, \mathbb{C})$  given by  $h \triangleright m = m_{(0)} < h, m_{(1)} >$ , such that  $h \triangleright 1 = \varepsilon(h)1$  and  $h \triangleright (mm') = (h_{(1)} \triangleright m)(h_{(2)} \triangleright m')$ .

From these properties, it follows that the generator  $K$  acts on  $M(3, \mathbb{C})$  by an automorphism, whereas  $X_{\pm}$  act as *twisted* derivations. On the basis of  $M(3, \mathbb{C})$  given by  $\tilde{x}^r\tilde{y}^s, r, s \in \{0, 1, 2\}$ , the action of generators  $X_{\pm}, K$  is given by:

$$\begin{aligned} K \triangleright (\tilde{x}^r\tilde{y}^s) &= q^{r-s} \tilde{x}^r\tilde{y}^s, \\ X_+ \triangleright (\tilde{x}^r\tilde{y}^s) &= \frac{q^r - q^{-r}}{q - q^{-1}} \tilde{x}^{r-1}\tilde{y}^{s+1}, \\ X_- \triangleright (\tilde{x}^r\tilde{y}^s) &= \frac{q^s - q^{-s}}{q - q^{-1}} \tilde{x}^{r+1}\tilde{y}^{s-1}, \end{aligned} \tag{21}$$

where the summations in exponents are meant modulo 3 and where repeated indices are not to be summed on. It is easy to see that there are three 3-dimensional invariant subspaces, generated respectively by  $\{\tilde{x}^2, \tilde{x}\tilde{y}, \tilde{y}^2\}$ ,  $\{\tilde{x}, \tilde{y}, \tilde{x}^2\tilde{y}^2\}$ ,  $\{1, \tilde{x}^2\tilde{y}, \tilde{x}\tilde{y}^2\}$ , such that on the first one  $\mathcal{H}$  acts irreducibly, whereas the last two are reducible indecomposable representation spaces.

Since  $M(3, \mathbb{C})$  is simple, the action of  $K$  is an inner automorphism, given in fact as the adjoint action of e.g.  $\tilde{K} = \tilde{x}^2 \tilde{y}^2$  and corresponding to the matrix  $U_1$  in eq.(10). In addition, the action of  $X_{\pm}$  as twisted derivations can be also expressed as a particular kind of internal operations. Indeed,  $M(3, \mathbb{C})$  can be viewed as a  $\mathbb{Z}_3$ -graded algebra with the grade of the monomials  $m = \tilde{x}^r \tilde{y}^s$  being given by  $|m| = r - s \bmod 3$ .

Then on any element  $m$  of grade  $|m|$  we have

$$\begin{aligned} X_+ \triangleright m &= \tilde{X}_+ m - q^{|m|} m \tilde{X}_+, \\ X_- \triangleright m &= q^{-|m|} \tilde{X}_- m - m \tilde{X}_-, \end{aligned} \quad (22)$$

where

$$\tilde{X}_+ = \frac{\tilde{x}^2 \tilde{y}}{q^{-1} - q} + C_+ \tilde{x}^2 \tilde{y}^2, \quad \tilde{X}_- = \frac{\tilde{x} \tilde{y}^2}{q - q^{-1}} + C_- \tilde{x}^2 \tilde{y}^2, \quad (23)$$

with  $C_+$ ,  $C_-$  being arbitrary constants.

Note that as elements of  $M(3, \mathbb{C})$ ,  $\tilde{K}$  and  $\tilde{X}_{\pm}$  do not obey exactly the same commutations rules of  $K$  and  $X_{\pm}$  in  $\mathcal{H}$ . For example, to get  $\tilde{X}_{\pm}^3 = 0$  one can set the constants  $C_+ = \frac{1}{q - q^{-1}}$ ,  $C_- = \frac{1}{q^{-1} - q}$ , but, with this choice one has  $\tilde{K} \tilde{X}_{\pm} \neq q^{\mp 2} \tilde{X}_{\pm} \tilde{K}$ , and so on.

We remark that by dualizing (15), we can extend this representation of  $\mathcal{H}$  on  $M(3, \mathbb{C})$  to a representation on  $\mathcal{A}$ , obtaining

$$h \triangleright (m + u + l) = m_{(0)} \langle h, m_{(1)} \rangle + (u + l) \varepsilon(h), \quad (24)$$

for any  $m \in M(3, \mathbb{C})$ ,  $u \in \mathbb{H}$  and  $l \in \mathbb{C}$ .

Also, in the same way we obtain from (16) a representation of  $\mathcal{H} \otimes \mathbb{C}(\mathbb{Z}_2)$  on  $\mathcal{A}$ , where  $\mathbb{C}(\mathbb{Z}_2)$  is the group algebra of  $\mathbb{Z}_2$ , which is in a natural duality with  $A(\mathbb{Z}_2)$  [M-S]. Explicitly, the action of a simple tensor  $h \otimes z$  in  $\mathcal{H} \otimes \mathbb{C}(\mathbb{Z}_2)$  on an element  $m + u + l$  turns out to be

$$h \otimes z \triangleright (m + u + l) = m_{(0)} \langle h, m_{(1)} \rangle \varepsilon(z) + u_{(0)} \langle z, u_{(1)} \rangle \varepsilon(h) + l \varepsilon(h) \varepsilon(z). \quad (25)$$

## 5 Further properties of $\mathcal{H}$ and $A(F)$

Using again duality we can compute two different commuting representations of  $\mathcal{H}$  on  $A(F)$ . One of them is given by  $\langle R(h)(\varphi), h' \rangle = \langle \varphi, h'h \rangle$ , or in Sweedler notation

$$h \triangleright \varphi = \varphi_{(1)} \langle h, \varphi_{(2)} \rangle. \quad (26)$$

Such a representation, which corresponds by duality [M-S] to the comultiplication in  $A(F)$ , makes  $A(F)$  a left- $\mathcal{H}$  module algebra. In table 2 of the Appendix we show the values of the action of the generators of  $\mathcal{H}$  via such representation on the basis of  $A(F)$ .

The other representation is given by  $\langle L(h)(\varphi), h' \rangle = \langle \varphi, S(h)h' \rangle$ , or in Sweedler notation

$$h \triangleright \varphi = \langle S(h), \varphi_{(0)} \rangle \varphi_{(1)}. \quad (27)$$

The representation  $L$  is such that  $h \triangleright (\varphi\psi) = (h_{(2)} \triangleright \varphi)(h_{(1)} \triangleright \psi)$ ,  $h \triangleright 1 = \varepsilon(h)$ , and corresponds to the right coaction of  $A(F)$  on itself given by  $\Delta_R = (id \otimes S) \circ \tau \circ \Delta$ , where  $\tau$  is the flip operator. In table 3 of the Appendix we present explicitly the action of generators.

Recall now that a left (resp. right) integral *on* a Hopf algebra  $H$  over a field  $k$  is a linear functional  $h : H \rightarrow k$  satisfying  $(Id \otimes h) \circ \Delta = 1_H \cdot h$  ( resp.  $(h \otimes Id) \circ \Delta = 1_H \cdot h$ ), whereas an element  $\lambda \in H$  is called a left (resp. right) integral *in*  $H$  if it verifies  $\alpha\lambda = \varepsilon(\alpha)\lambda$ , or, respectively,  $\lambda\alpha = \varepsilon(\alpha)\lambda$ , for any  $\alpha \in H$ . For finite dimensional Hopf algebras, integrals *in*  $H$  are nothing but integrals *on* the dual  $H^*$  and both the spaces of left and right integrals are one dimensional [S-M]. A Hopf algebra  $H$  is called *unimodular* if there are left and right integrals on  $H$  which coincide. An integral on unimodular  $H$  is called a *Haar measure* iff it is *normalized*, i.e.  $h(1) = 1$ .

In our case we have that the Hopf algebra  $A(F)$  is unimodular with the left and right integrals being given by  $h = C(\tilde{b}^2\tilde{c}^2)^*$ , i.e.  $h(\tilde{b}^2\tilde{c}^2) = C \in k$ ,  $h(\tilde{a}^p\tilde{b}^r\tilde{c}^s) = 0$  if  $(p, r, s) \neq (0, 2, 2)$  [DHS]. In terms of the basis of  $\mathcal{H} = A(F)^*$ ,  $h = CX_-^2X_+^2(1 + K + K^2)$ . Being  $h$  not normalizable, it follows that there is no Haar measure on the Hopf algebra  $A(F)$  and, consequently,  $F$  is not a compact matrix quantum group in the sense of [W-S].

As regards integrals *in*  $A(F)$ , it is easy to see that the element  $\lambda_L = (1 + \tilde{a} + \tilde{a}^2)\tilde{b}^2\tilde{c}^2$  is a left integral and that the element  $\lambda_R = \tilde{b}^2\tilde{c}^2(1 + \tilde{a} + \tilde{a}^2)$  is a right one. Thought as integrals *on*  $\mathcal{H}$ ,  $\lambda_L = (X_-^2X_+^2K)^*$  and  $\lambda_R = (X_-^2X_+^2K^2)^*$ . Thus in this case left and right integral are not proportional. It is evident, now, that, as stated in Proposition 7 in [LS], there exist (left and right) integrals in and on  $A(F)$  and  $\mathcal{H}$  such that  $\langle h, \lambda \rangle = 1$ .

In addition, by Theorem 5.18 in [S-M], the property  $\varepsilon(\lambda_L) = \varepsilon(h) = 0$  assures us that both  $A(F)$  and  $H$  are neither semisimple nor cosemisimple.

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## Appendix: Explicit Duality and Representations

$\langle \cdot   \cdot \rangle$	$\tilde{b}^2 \tilde{c}^2$	$\tilde{a} \tilde{b}^2 \tilde{c}^2$	$\tilde{d} \tilde{b}^2 \tilde{c}^2$	$\tilde{b} \tilde{c}$	$\tilde{a} \tilde{b} \tilde{c}$	$\tilde{d} \tilde{b} \tilde{c}$	1	$\tilde{a}$	$\tilde{d}$
$X_-^2 X_+^2$	1	$q^2$	$q$	0	$q$	0	0	0	0
$X_-^2 X_+^2 K$	1	1	1	0	$q^2$	0	0	0	0
$X_-^2 X_+^2 K^2$	1	$q$	$q^2$	0	1	0	0	0	0
$X_- X_+$	0	0	0	1	$q$	$q^2$	0	1	0
$X_- X_+ K$	0	0	0	1	$q^2$	$q$	0	$q$	0
$X_- X_+ K^2$	0	0	0	1	1	1	0	$q^2$	0
1	0	0	0	0	0	0	1	1	1
$K$	0	0	0	0	0	0	1	$q$	$q^2$
$K^2$	0	0	0	0	0	0	1	$q^2$	$q$

$\langle \cdot   \cdot \rangle$	$\tilde{c}^2$	$\tilde{a} \tilde{c}^2$	$\tilde{d} \tilde{c}^2$
$X_+^2$	-1	$-q^2$	$-q$
$X_+^2 K$	$-q^2$	$-q^2$	$-q^2$
$X_+^2 K^2$	$-q$	$-q^2$	-1

$\langle \cdot   \cdot \rangle$	$\tilde{b}^2$	$\tilde{a} \tilde{b}^2$	$\tilde{d} \tilde{b}^2$
$X_-^2$	-1	-1	-1
$X_-^2 K$	$-q$	$-q^2$	-1
$X_-^2 K^2$	$-q^2$	$-q$	-1

$\langle \cdot   \cdot \rangle$	$\tilde{b}^2 \tilde{c}$	$\tilde{a} \tilde{b}^2 \tilde{c}$	$\tilde{d} \tilde{b}^2 \tilde{c}$	$\tilde{b}$	$\tilde{a} \tilde{b}$	$\tilde{d} \tilde{b}$
$X_-^2 X_+$	-1	$-q$	$-q^2$	0	-1	0
$X_-^2 X_+ K$	$-q^2$	$-q$	-1	0	-1	0
$X_-^2 X_+ K^2$	$-q$	$-q$	$-q$	0	-1	0
$X_-$	0	0	0	1	1	1
$X_- K$	0	0	0	$q^2$	1	$q$
$X_- K^2$	0	0	0	$q$	1	$q^2$

$\langle \cdot   \cdot \rangle$	$\tilde{b} \tilde{c}^2$	$\tilde{a} \tilde{b} \tilde{c}^2$	$\tilde{d} \tilde{b} \tilde{c}^2$	$\tilde{c}$	$\tilde{a} \tilde{c}$	$\tilde{d} \tilde{c}$
$X_- X_+^2$	-1	$-q^2$	$-q$	0	$-q$	0
$X_- X_+^2 K$	$-q$	$-q$	$-q$	0	-1	0
$X_- X_+^2 K^2$	$-q^2$	-1	$-q$	0	$-q^2$	0
$X_+$	0	0	0	1	$q$	$q^2$
$X_+ K$	0	0	0	$q$	1	$q^2$
$X_+ K^2$	0	0	0	$q^2$	$q^2$	$q^2$

Table 1: Diagonal blocks in the pairing of  $\mathcal{H}$  and  $A(F)$

R(h)	$X_-$	$X_+$	$K$
1	0	0	1
$\tilde{a}$	0	$\tilde{b}$	$q\tilde{a}$
$\tilde{d}$	$\tilde{c}$	0	$q^2\tilde{d}$
$\tilde{b}$	$\tilde{a}$	0	$q^2\tilde{b}$
$\tilde{a}\tilde{b}$	$\tilde{d} - q\tilde{d}\tilde{b}\tilde{c} + q^2\tilde{d}\tilde{b}^2\tilde{c}^2$	$\tilde{b}^2$	$\tilde{a}\tilde{b}$
$\tilde{d}\tilde{b}$	$1 - \tilde{b}\tilde{c}$	0	$q\tilde{d}\tilde{b}$
$\tilde{b}^2$	$-\tilde{a}\tilde{b}$	0	$q\tilde{b}^2$
$\tilde{a}\tilde{b}^2$	$-\tilde{d}\tilde{b} + q\tilde{d}\tilde{b}^2\tilde{c}$	0	$q^2\tilde{a}\tilde{b}^2$
$\tilde{d}\tilde{b}^2$	$-\tilde{b}$	0	$\tilde{d}\tilde{b}^2$
$\tilde{c}$	0	$\tilde{d}$	$q\tilde{c}$
$\tilde{a}\tilde{c}$	0	$q - q\tilde{b}\tilde{c}$	$q^2\tilde{a}\tilde{c}$
$\tilde{d}\tilde{c}$	$q^2\tilde{c}^2$	$q^2\tilde{a} - q\tilde{a}\tilde{b}\tilde{c} + \tilde{a}\tilde{b}^2\tilde{c}^2$	$\tilde{d}\tilde{c}$
$\tilde{b}\tilde{c}$	$q^2\tilde{a}\tilde{c}$	$\tilde{d}\tilde{b}$	$\tilde{b}\tilde{c}$
$\tilde{a}\tilde{b}\tilde{c}$	$q^2\tilde{d}\tilde{c} - \tilde{d}\tilde{b}\tilde{c}^2$	$q\tilde{b} - q\tilde{b}^2\tilde{c}$	$q\tilde{a}\tilde{b}\tilde{c}$
$\tilde{d}\tilde{b}\tilde{c}$	$q^2\tilde{c} + \tilde{b}\tilde{c}^2 + q\tilde{b}\tilde{c}^2$	$q^2\tilde{a}\tilde{b} - q\tilde{a}\tilde{b}^2\tilde{c}$	$q^2\tilde{d}\tilde{b}\tilde{c}$
$\tilde{b}^2\tilde{c}$	$-q^2\tilde{a}\tilde{b}\tilde{c}$	$\tilde{d}\tilde{b}^2$	$q^2\tilde{b}^2\tilde{c}$
$\tilde{a}\tilde{b}^2\tilde{c}$	$-q^2\tilde{d}\tilde{b}\tilde{c} + \tilde{d}\tilde{b}^2\tilde{c}^2$	$q\tilde{b}^2$	$\tilde{a}\tilde{b}^2\tilde{c}$
$\tilde{d}\tilde{b}^2\tilde{c}$	$-q^2\tilde{b}\tilde{c}$	$q^2\tilde{a}\tilde{b}^2$	$q\tilde{d}\tilde{b}^2\tilde{c}$
$\tilde{c}^2$	0	$-q\tilde{d}\tilde{c}$	$q^2\tilde{c}^2$
$\tilde{a}\tilde{c}^2$	0	$-q^2\tilde{c}$	$\tilde{a}\tilde{c}^2$
$\tilde{d}\tilde{c}^2$	0	$-\tilde{a}\tilde{c} + q^2\tilde{a}\tilde{b}\tilde{c}^2$	$q\tilde{d}\tilde{c}^2$
$\tilde{b}\tilde{c}^2$	$q\tilde{a}\tilde{c}^2$	$-q\tilde{d}\tilde{b}\tilde{c}$	$q\tilde{b}\tilde{c}^2$
$\tilde{a}\tilde{b}\tilde{c}^2$	$q\tilde{d}\tilde{c}^2$	$-q^2\tilde{b}\tilde{c}$	$q^2\tilde{a}\tilde{b}\tilde{c}^2$
$\tilde{d}\tilde{b}\tilde{c}^2$	$q\tilde{c}^2$	$-\tilde{a}\tilde{b}\tilde{c} + q^2\tilde{a}\tilde{b}^2\tilde{c}^2$	$\tilde{d}\tilde{b}\tilde{c}^2$
$\tilde{b}^2\tilde{c}^2$	$-q\tilde{a}\tilde{b}\tilde{c}^2$	$-q\tilde{d}\tilde{b}^2\tilde{c}$	$\tilde{b}^2\tilde{c}^2$
$\tilde{a}\tilde{b}^2\tilde{c}^2$	$-q\tilde{d}\tilde{b}\tilde{c}^2$	$-q^2\tilde{b}^2\tilde{c}$	$q\tilde{a}\tilde{b}^2\tilde{c}^2$
$\tilde{d}\tilde{b}^2\tilde{c}^2$	$-q\tilde{b}\tilde{c}^2$	$-\tilde{a}\tilde{b}^2\tilde{c}$	$q^2\tilde{d}\tilde{b}^2\tilde{c}^2$

Table 2: Action of the generators of  $\mathcal{H}$  via the representation  $R$

L(h)	$X_-$	$X_+$	$K$
1	0	0	1
$\tilde{a}$	$-q^2\tilde{c}$	0	$q^2\tilde{a}$
$\tilde{d}$	0	$-q\tilde{b}$	$q\tilde{d}$
$\tilde{b}$	$-q^2\tilde{d}$	0	$q^2\tilde{b}$
$\tilde{a}\tilde{b}$	$-1 + \tilde{b}\tilde{c}$	$q$	$\tilde{a}\tilde{b}$
$\tilde{d}\tilde{b}$	$-q\tilde{a} + \tilde{a}\tilde{b}\tilde{c} - q^2\tilde{a}\tilde{b}^2\tilde{c}^2$	$-\tilde{b}^2$	$\tilde{d}\tilde{b}$
$\tilde{b}^2$	$\tilde{d}\tilde{b}$	0	$q\tilde{b}^2$
$\tilde{a}\tilde{b}^2$	$q\tilde{b}$	0	$\tilde{a}\tilde{b}^2$
$\tilde{d}\tilde{b}^2$	$q^2\tilde{a}\tilde{b} - q\tilde{a}\tilde{b}^2\tilde{c}$	0	$q^2\tilde{d}\tilde{b}^2$
$\tilde{c}$	0	$-q\tilde{a}$	$q\tilde{c}$
$\tilde{a}\tilde{c}$	$-q^2\tilde{c}^2$	$-q\tilde{d} + q^2\tilde{d}\tilde{b}\tilde{c} - \tilde{d}\tilde{b}^2\tilde{c}^2$	$\tilde{a}\tilde{c}$
$\tilde{d}\tilde{c}$	0	$-q + q\tilde{b}\tilde{c}$	$q^2\tilde{d}\tilde{c}$
$\tilde{b}\tilde{c}$	$-q^2\tilde{d}\tilde{c}$	$-\tilde{a}\tilde{b}$	$\tilde{b}\tilde{c}$
$\tilde{a}\tilde{b}\tilde{c}$	$-\tilde{c} + \tilde{b}\tilde{c}^2$	$-\tilde{d}\tilde{b} + q\tilde{d}\tilde{b}^2\tilde{c}$	$q^2\tilde{a}\tilde{b}\tilde{c}$
$\tilde{d}\tilde{b}\tilde{c}$	$-q\tilde{a}\tilde{c} + \tilde{a}\tilde{b}\tilde{c}^2$	$-\tilde{b} + \tilde{b}^2\tilde{c}$	$q\tilde{d}\tilde{b}\tilde{c}$
$\tilde{b}^2\tilde{c}$	$\tilde{d}\tilde{b}\tilde{c}$	$-q^2\tilde{a}\tilde{b}^2$	$q^2\tilde{b}^2\tilde{c}$
$\tilde{a}\tilde{b}^2\tilde{c}$	$q\tilde{b}\tilde{c}$	$-q^2\tilde{d}\tilde{b}^2$	$q\tilde{a}\tilde{b}^2\tilde{c}$
$\tilde{d}\tilde{b}^2\tilde{c}$	$q^2\tilde{a}\tilde{b}\tilde{c} - q\tilde{a}\tilde{b}^2\tilde{c}^2$	$-q^2\tilde{b}^2$	$\tilde{d}\tilde{b}^2\tilde{c}$
$\tilde{c}^2$	0	$q\tilde{a}\tilde{c}$	$q^2\tilde{c}^2$
$\tilde{a}\tilde{c}^2$	0	$q\tilde{d}\tilde{c} - q^2\tilde{d}\tilde{b}\tilde{c}^2$	$q\tilde{a}\tilde{c}^2$
$\tilde{d}\tilde{c}^2$	0	$q\tilde{c}$	$\tilde{d}\tilde{c}^2$
$\tilde{b}\tilde{c}^2$	$-q^2\tilde{d}\tilde{c}^2$	$\tilde{a}\tilde{b}\tilde{c}$	$q\tilde{b}\tilde{c}^2$
$\tilde{a}\tilde{b}\tilde{c}^2$	$-\tilde{c}^2$	$\tilde{d}\tilde{b}\tilde{c} - q\tilde{d}\tilde{b}^2\tilde{c}^2$	$\tilde{a}\tilde{b}\tilde{c}^2$
$\tilde{d}\tilde{b}\tilde{c}^2$	$-q\tilde{a}\tilde{c}^2$	$\tilde{b}\tilde{c}$	$q^2\tilde{d}\tilde{b}\tilde{c}^2$
$\tilde{b}^2\tilde{c}^2$	$\tilde{d}\tilde{b}\tilde{c}^2$	$q^2\tilde{a}\tilde{b}^2\tilde{c}$	$\tilde{b}^2\tilde{c}^2$
$\tilde{a}\tilde{b}^2\tilde{c}^2$	$q\tilde{b}\tilde{c}^2$	$q^2\tilde{d}\tilde{b}^2\tilde{c}$	$q^2\tilde{a}\tilde{b}^2\tilde{c}^2$
$\tilde{d}\tilde{b}^2\tilde{c}^2$	$q^2\tilde{a}\tilde{b}\tilde{c}^2$	$q^2\tilde{b}^2\tilde{c}$	$q\tilde{d}\tilde{b}^2\tilde{c}^2$

Table 3: Action of the generators of  $\mathcal{H}$  via the representation  $L$



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